MODELING DEFAULT-FREE BOND YIELD CURVES

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The term structure of interest rates has received increased attention in recent years. The underlying reason for focusing on the relationship among the yields on default-free securities that differ in their maturity is that such structure reflects information concerning the future course of events. Such information may be useful to economists, policymakers, and market practitioners, because it appears to predict real economic activity. (See, for example, Estrella and Hardouvellis [1991], Harvey [1988, 1989], Laurent [1988], Chen [1989], Stock and Watson [1989], and Kazemi [1988].)

The earlier empirical literature on the structure of interest rates was mainly concerned with the informative content of forward rates. These studies were motivated by three theories of term structure: the Expectations Hypothesis, the Liquidity Premium Hypothesis, and the Preferred Habitat Hypothesis. Abken [1990] provides a detailed discussion of these theories, while Hardouvellis [1988] and Estrella and Hardouvellis [1991] review some of the empirical studies.

Many alternative estimation methods for the yield curve have appeared in the literature over the years. They range from the early eye-balling of curve fitting to the lower boundary of secondary market yields on U.S. corporate bonds by Durand [1942], subsequently adapted by the U.S. Treasury, to the more objective regression-based approaches. These parametric models include the studies of Chambers, Carleton, and Waldman [1984], Cohen, Cramer, and Waugh [1966], Dobson [1978], Heller and Khan [1979], Echols and Elliott [1976], Fisher [1966], McCulloch [1971, 1975], Nelson and Siegel [1987], Ogden

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[1987], Power [1991], Shea [1982, 1984, 1985], and Vasicek and Fong [1982].[1]

This article combines the methodologies of Nelson and Siegel [1987], Siegel and Nelson [1988], and Cox, Ingersoll, and Ross [1985] to obtain a testable form of the term to maturity structure of long-term interest rates. Essentially, the proposed model adopts the methodology suggested by Siegel and Nelson [1988], who noted that, under general conditions, the long-term yields approach their asymptotic value at a rate proportional to the reciprocal of the time to maturity. Thus, the yield on a long-term default-free bond is equal to its asymptotic value adjusted for the risk premium pertinent to the maturity of the bond.[2]

The model uses the asymptotic value of the yield derived by Cox, Ingersoll, and Ross [1985] and adjusts it for its specific maturity employing a curve-fitting approach suggested by Nelson and Siegel [1987]. The new parsimonious model predicts default-free bond prices with significantly greater accuracy than the original model suggested by Nelson and Siegel [1987].

THE MODEL

Siegel and Nelson show that "if forward interest rates do behave in a stable way as a function of maturity, then a rather weak condition on that stability implies that yield curves *must* approach an asymptote at a rate that is proportional to the reciprocal of maturity" [1988, p. 106]. In deriving their general form of modeling yield curves, Siegel and Nelson require two assumptions concerning the convergence of the forward rate, r(m), to its asymptotic value.

$$r(\infty) = \lim_{m\to\infty} r(m) \qquad \text{exists and is finite} \qquad \text{A.1}$$

$$\lim_{m\to\infty} \int_0^m [r(t) - r(\infty)]dt = c \qquad \text{exists and is finite} \qquad \text{A.2}$$

Assumption A.1 states that the forward rate has a finite limiting value. Assumption A.2 requires that the forward rate approach this asymptotic value quickly enough. Under Assumptions A.1 and A.2 and according to McCulloch's [1971] representation of the yield and forward rates[3]

$$\text{Yield:} \qquad R(m) = \frac{1}{m}\int_0^m r(t)dt \qquad (1)$$

$$\text{Forward Rate: } r(m) = R(m) + mR'(m) \qquad (2)$$

it follows that the asymptotic value of the yield exists and is equal to the asymptotic forward rate, or

$$R(\infty) = \lim_{m\to\infty} R(m) = r(\infty) \qquad (3)$$

The final form of the yield becomes

$$R(m) = R(\infty) + \frac{c}{m} + \int(m) \qquad (4)$$

where $\int(m)$ is a remainder term that tends to zero at a rate faster than 1/m as $m\to\infty$ (that is, $m\int(m)\to 0$) (see Siegel and Nelson [1988, p. 107] for a formal proof). Equation (4), in the words of Siegel and Nelson, states that "the yield R(m) tends toward its asymptotic value $R(\infty)$ with decay dominated by c/m, with higher-order (faster) decay terms represented by the remainder term $\int(m)$" [1988, p. 107].

Equation (4) is very general and deserves further elaboration. First, because it is concerned with the asymptotic value of the yield, it is more appropriate for the long-term yield structure. Thus, it is expected to produce better results when applied to longer-maturity as opposed to shorter-maturity default-free securities.

Second, it is an approximation, especially if we drop the remainder term $\int(m)$. That is,

$$R(m) \approx R(\infty) + \frac{c}{m} \qquad (4')$$

Equation (4′) may be testable if $R(\infty)$ and c can be estimated from existing models of the term structure of interest rates. For example, Cox, Ingersoll, and Ross [1985] (CIR) give an expression of $R(\infty)$ assuming the price of the discount bond as a function of time to maturity, m. The CIR limiting value of the yield-to-maturity, $R(\infty)$, is

$$R(\infty) = \left(\frac{2\kappa}{\gamma + \kappa + \lambda}\right)\theta \qquad (5)$$

where κ represents the speed of the adjustment (mean reversion if $\kappa > 0$) of the interest rate toward its long-term value θ, λ is the market price of instantaneous interest rate risk, and $\gamma = [(\kappa + \lambda)^2 + 2\sigma^2]^{1/2}$ with σ^2 being the variance of the spot rate. As Brown and Dybvig [1986] point out, Equation (5) essentially corresponds to a single yield curve for discount bonds.

Our task is to derive an approximation for the limit of the area between the forward rate curve and its asymptotic value, or c. For this purpose, we need an expression for the forward rate. Following Nelson and Siegel [1987], let us denote the instantaneous forward rate at maturity m, r(m), by the solution to a second-order differential equation with real and equal roots.

$$r(m) = \theta + \beta_1 e^{-m/\tau} + \beta_2 \left[(m/\tau)e^{-m/\tau}\right] \qquad (6)$$

The first term, θ, represents the long-term value of the interest rate, as seen in the CIR model.[4] The second and third terms indicate the contribution of the short-term and medium-term components. The time constant τ determines the rate of decay in the short- and medium-term variables so as to make them more suitable to fit curvatures at different maturities.

For example, small values of τ result in rapid decay in the predictor variables and therefore will be suitable for curvature at low maturities. By the same token, large values of τ produce slow decay in the predictor variables and will be able to fit curvature over longer maturities.

As Nelson and Siegel [1987] note, the last two terms of Equation (6) represent a Laquerre function and together with asymptote θ may be used to approximate all possible shapes of the yield curve; namely, humped, S, or monotonic. Following McCulloch's [1971] definition of the yield as an average of the forward rates r(.), as given in (1), we integrate (6) from zero to m and divide by m to obtain:

$$R(m) = \vartheta + (\beta_1 + \beta_2)\left[\frac{1 - e^{-m/\tau}}{m/\tau}\right] - \beta_2 e^{-m/\tau} \qquad (7)$$

which is also linear in coefficients, given τ.

It is obvious from Equation (7) that the choice of τ will affect the values of the parameters θ, β_1, and β_2. To assume that an arbitrary constant value of τ will fit all yield curves over time is tantamount to the statement that the rate of decay in the yield curve remains practically unchanged over time. Thus, a question arises as to the optimal value of τ in (7) across the maturity spectrum of default-free securities.

Assuming the Nelson and Siegel [1987] approximation of the yield curve as given in (7), it can be shown that for long maturities the decay term c/m in Equation (4) may be approximated by the second term, $(\beta_1 + \beta_2)[1 - e^{-m/\tau}]/(m/\tau)$, in Equation (7).[5] Nelson and Siegel define the optimal value of τ, τ^*, as the value that maximizes the R^2 in the regression representation of (7) fitted to Treasury bill yields. When they used Equation (7) to forecast the price of a long-term bond, however, they discovered that the predicted bond price was overshooting the actual, although their correlation was quite high.

The intuitive reason for the inability of the model to predict yields or prices at maturities beyond the range of the sample lies with the risk premium associated with each default-free bond, which may not be described by maturity differentials alone. This is precisely the contribution of the CIR general equilibrium model. Because the decaying rate c/m in (4′) is reasonably quick, by Assumption A.2 it follows that the mispricing of bonds with long maturities using Equation (7) must be largely due to misspecification of the asymptotic value, θ.

Thus, adjusting θ to reflect the risk premium associated with each maturity, as suggested by the CIR model, and approximating c/m by the second term in (7), Equation (4) may be rewritten as:

$$R(m) = \left(\frac{2\kappa}{\gamma + \kappa + \lambda}\right)\vartheta + (\beta_1 + \beta_2)\left[\frac{1 - e^{-m/\tau^*}}{m/\tau^*}\right] + \int(m) \qquad (8)$$

where all terms are as defined earlier, and τ^* indicates the optimal value of the time constant τ.

Equation (8) is the fundamental pricing equation of this article. Basically, it decomposes the interest rate risk premium into two components. The first part of the risk premium depends on risk-return preference parameters that cannot be determined from the current level of interest and cannot be priced from the term structure of T-bills. The remaining risk premium is related to the curvature (rate of decay) of the yield curve, and it can be estimated from the term structure of T-bills.

Of course, when Equation (8) is applied to T-bills, it reduces to (7), as the choice of τ^* captures almost all uncertainty. The difficulty with this equation is that the CIR parameters κ, λ, and γ (or σ^2) are not directly available. They can, however, be estimated by solving a system of simultaneous equations.

Chen and Scott [1990] estimated these parameters using maximum likelihood based on both time series and cross-sectional data on bond prices. Their sample covers the period of January 1980 through December 1988, and consists of Thursday prices for Treasury bills, notes, and bonds. Their estimates are as follows, with standard errors in parentheses:

κ	=	0.9287	(0.1018)
θ	=	0.0926	(0.0095)
σ	=	0.1106	(0.0019)
λ	=	-0.0596	(0.0958)

These estimates imply a limiting value for the yield during that period: $R(\infty) = 0.0981$.

In an earlier study, Brown and Dybvig [1986] estimated the limiting value of the yield, among other variables, also using maximum likelihood and data on Treasury bills, notes, and bonds. Their period of investigation runs from December 1952 to December 1983. Their findings concerning the long-term yield implied by the model are significantly lower for the premium issues and significantly higher for the discount issues. They attribute this finding to the existence of taxes, not considered by the model.

If we assume that the CIR parameters κ, λ, and σ^2 change through time so as to keep the ratio $[2\kappa/(\gamma + \kappa + \lambda)]$ either constant or near constant, we can approximate the current period's ratio with its lagged value, i.e., $[2\kappa/(\gamma + \kappa + \lambda)]_t \approx [2\kappa/(\gamma + \kappa + \lambda)]_{t+\delta}$ where δ represents a short period of time, such as one day. Following the methodology adopted by Nelson and Siegel [1987], we can then use Equation (8) to forecast long-term yields from the term structure implied by the Treasury bills.

In that case, the ratio $[2\kappa/(\gamma + \kappa + \lambda)]$ becomes approximately equal to one when Equation (8) is fitted to Treasury bills. As we would intuitively expect, because the return on Treasury bills is certain, they should bear no risk in the CIR sense.[6]

EMPIRICAL EVIDENCE

Here we use the methodology described to forecast the prices of three bonds with different maturities during 1988. Essentially, we fit Equation (7) to weekly data on U.S. Treasury bills to obtain estimates of the required parameters, and then use these parameters to test Equation (8).

The raw data on Treasury bills were obtained from the *Wall Street Journal* for every Monday and Thursday during 1988. The data consist of bid and asked discounts and bond-equivalent yields for the bills in each maturity date outstanding as of the close of each trading day. Typically, the first twenty-five to twenty-six maturities are in increments of seven days, while the remaining six to seven maturities are in increments of twenty-eight days. Occasionally, there may also be a one-year bill traded. The bid and ask discounts are calculated and reported on the basis of a 360-day year and are on a simple interest basis. We can calculate the price of a bill from the asked discount quote, as follows

$$P(m) = 1 - \frac{am}{360} \tag{9}$$

where P(m) is the price of the bill with maturity m, m is the time to maturity (in days), and a is the asked discount. The number of days to maturity, m, is calculated from the delivery date until the maturity date. The Federal Reserve assumes a two-business day settlement period, so the delivery date for a Thursday transaction is the following Monday.

Once we obtain bill prices, we can then calculate implied continuously compounded rates of return, R(m), as follows

$$R(m) = -\left(\frac{365}{m}\right)\ln P(m) \qquad (10)$$

Equation (10) provides the yields required to fit Equation (7). Observations on the first two maturities were omitted because the yields were inconsistent with the remaining data.[7] In search of the optimal τ, the model is fitted twenty times daily (for every Monday and Thursday during 1988), varying the value of τ from 10 to 200 in increments of 10. The optimal τ, τ^*, is identified as the one with the best model fit, or, equivalently, the highest R^2.

Exhibits 1 and 2 present the results of the second-order model fitted over forty-eight Mondays and fifty-one Thursdays during 1988. The exhibits give the date, the optimal τ, the standard deviation of residuals in basis points (hundredths of a percent) at the optimal, and median values of τ, the R^2, and the yields at both the optimal and median τ. The high values of R^2 indicate the appropriateness of the model fit represented by Equation (7).

Although the median value of τ was 40 for both days of the week, it is interesting to note that the last quarter of 1988 resulted in excessive boundary values of $\tau = 10$. As explained earlier, this may be due to the rapid decay of the yield curves associated with that period.[8] The model fit yielded four upper-boundary solutions ($\tau = 200$) for Mondays and three for Thursdays, indicating the existence of a curvature in the corresponding yield curves at longer maturities.

EXHIBIT 1 ■ Measures of Model Fit ■ Mondays 1988

		Second-Order Model				
Date	Optimal tau	SD at Best tau	SD at tau = 40	R^2	Yield at Best tau	Yield at tau = 40
01/04/88	10	87.49	86.46	0.95	6.83	6.80
01/11/88	40	62.50	62.50	0.99	6.86	6.86
01/18/88	40	35.09	35.09	0.98	6.76	6.76
01/25/88	50	41.74	41.72	0.94	6.60	6.59
02/01/88	200	32.25	32.19	0.95	6.49	6.47
02/08/88	10	41.99	41.58	0.89	6.15	6.26
02/15/88	200	38.07	38.02	0.98	6.52	6.51
02/22/88	80	34.66	34.58	0.97	6.39	6.36
02/29/88	120	32.58	32.55	0.96	6.34	6.30
03/07/88	110	32.84	32.82	0.95	6.44	6.40
03/14/88	170	30.50	30.18	0.92	6.43	6.35
03/21/88	90	32.66	32.48	0.94	6.45	6.38
03/28/88	50	27.60	27.49	0.84	6.57	6.53
04/04/88	40	27.98	27.98	0.71	6.70	6.70
04/11/88	140	28.31	27.91	0.84	6.71	6.62
04/18/88	50	35.88	35.85	0.98	6.78	6.76
04/25/88	50	35.58	35.57	0.97	6.78	6.75
05/02/88	110	37.77	37.66	0.97	6.92	6.86
05/09/88	80	33.97	33.85	0.95	7.00	6.95
05/16/88	70	38.90	38.86	0.99	6.99	6.97
05/23/88	90	39.45	39.34	0.97	7.18	7.13
06/06/88	50	41.61	41.60	0.97	7.17	7.16
06/13/88	70	30.94	30.88	0.96	7.10	7.07
06/20/88	30	32.39	32.37	0.87	7.29	7.33
06/27/88	40	36.11	36.11	0.94	7.17	7.17

EXHIBIT 1 ■ Continued

		Second-Order Model				
Date	Optimal tau	SD at Best tau	SD at tau = 40	R^2	Yield at Best tau	Yield at tau = 40
07/11/88	60	37.63	37.60	0.98	7.41	7.39
07/18/88	10	42.03	41.34	0.97	7.39	7.50
07/25/88	40	35.53	35.53	0.97	7.43	7.43
08/01/88	10	47.21	46.86	0.88	7.44	7.44
08/08/88	20	55.62	55.21	0.90	7.57	7.60
08/15/88	10	54.39	53.74	0.85	7.81	7.88
08/22/88	10	46.96	46.52	0.73	7.65	7.58
08/29/88	60	31.82	31.76	0.76	7.94	7.92
09/12/88	200	19.73	19.73	0.90	7.78	7.78
09/19/88	200	25.49	25.45	0.92	7.71	7.74
09/26/88	30	27.23	27.23	0.83	7.85	7.86
10/03/88	30	29.33	29.31	0.95	7.80	7.80
10/10/88	10	20.67	20.63	0.91	7.69	7.71
10/17/88	10	25.65	25.49	0.83	7.73	7.71
10/24/88	10	26.22	25.27	0.87	7.80	7.78
10/31/88	10	33.14	32.89	0.88	7.74	7.73
11/07/88	10	31.23	30.61	0.76	8.00	8.00
11/14/88	10	42.13	41.00	0.85	8.13	8.00
11/21/88	10	42.52	38.09	0.92	8.23	8.12
11/28/88	10	57.07	55.84	0.81	8.45	8.30
12/05/88	10	71.65	70.22	0.80	8.80	8.59
12/19/88	10	38.19	37.87	0.88	8.72	8.71
12/26/88	30	70.22	69.97	0.88	8.51	8.45

Although the optimal values of τ vary considerably, as shown in Exhibits 1 and 2, rather little precision of fit is lost if we impose the restriction of $\tau = 40$, the median value for all data sets. This is shown in Exhibits 3 and 4. These figures plot the actual and estimated yields using the median value of $\tau = 40$. As is evident from the values of the standard deviations in Exhibits 1 and 2, little may be gained in practice by fitting τ to each data set individually.

To explore the predictive ability of the Nelson and Siegel model, we use the parameters obtained from fitting Equation (7) to forecast the yields of three U.S. Treasury bonds representing three different maturities. The first bond chosen to represent the short-term case is the 8 1/4% coupon U.S. Treasury bond maturing in December 1991. The intermediate-term maturity is represented by the 11 7/8% coupon bond maturing in November 2003, and the long-term case is represented by the 9 7/8% coupon bond maturing in November 2015.

In all cases, the price of the bonds is computed on the basis of the principle that a bond may be thought of as a bundle of bills consisting of the coupons with maturities spaced at six-month intervals and the face value payment at the maturity date of the bond. By summing these discounted semiannual coupon payments and the discounted final payment of the bond, we obtain its price.[9] The resulting bond price can be compared with the quoted price of the bond.

Exhibits 5 through 7 compare the actual (quoted) bond price to the predicted bond price for the short-term bond, the intermediate bond, and the long-term bond. The predicted values are based on the median value of $\tau = 40$ and the measures of the model fit for Mondays of 1988, as shown in Exhibit 1. These figures suggest that the proposed Nelson-Siegel second-order yield curve model may not be satisfactory in predicting bond prices of maturities beyond the range of the sample used to fit it. Specifically, the model overprices all three bonds.[10]

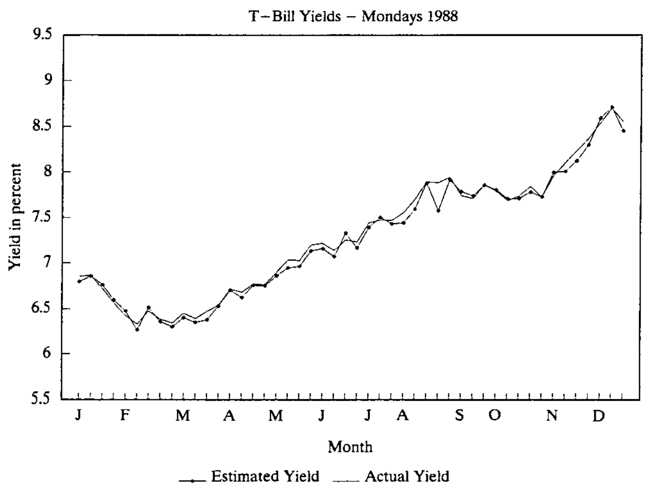
EXHIBIT 2 ■ Measures of Model Fit ■ Thursdays 1988

		Second-Order Model				
Date	Optimal tau	SD at Best tau	SD at tau = 40	R^2	Yield at Best tau	Yield at tau = 40
07-Jan	10	72.01	71.93	0.99	6.76	6.82
14-Jan	50	39.73	39.72	0.97	6.87	6.84
21-Jan	60	41.11	41.10	0.95	6.65	6.64
28-Jan	80	38.10	38.04	0.96	6.52	6.49
04-Feb	10	43.71	43.41	0.81	6.32	6.40
11-Feb	120	31.36	31.17	0.96	6.38	6.32
18-Feb	40	42.93	42.93	0.95	6.39	6.39
25-Feb	100	35.23	35.16	0.96	6.40	6.35
03-Mar	40	33.99	33.99	0.91	6.34	6.34
10-Mar	170	29.95	29.69	0.94	6.44	6.36
17-Mar	70	31.98	31.90	0.91	6.29	6.25
24-Mar	70	26.68	26.51	0.78	6.59	6.52
31-Mar	40	26.21	26.21	0.69	6.52	6.52
07-Apr	200	28.74	28.14	0.85	6.76	6.63
14-Apr	50	32.23	32.19	0.98	6.64	6.61
21-Apr	50	37.55	37.55	0.97	6.80	6.77
28-Apr	110	36.15	36.04	0.96	6.83	6.77
05-May	90	34.83	34.70	0.96	6.89	6.84
12-May	70	39.33	39.30	0.98	7.00	6.98
19-May	200	38.82	38.79	0.98	7.10	7.07
26-May	40	45.17	45.17	0.92	7.23	7.23
02-Jun	60	41.03	41.01	0.98	7.27	7.25
09-Jun	60	32.51	32.48	0.96	7.09	7.07
16-Jun	40	29.12	29.12	0.96	7.12	7.12
23-Jun	30	38.06	38.04	0.85	7.10	7.11
30-Jun	30	33.37	33.34	0.90	7.08	7.10
07-Jul	50	36.79	36.78	0.98	7.21	7.20
14-Jul	10	37.70	37.43	0.97	7.38	7.47
21-Jul	40	36.15	36.15	0.96	7.44	7.44
28-Jul	20	34.78	34.67	0.93	7.44	7.45
04-Aug	40	35.66	35.66	0.96	7.47	7.47
11-Aug	10	55.70	55.13	0.86	7.84	7.88
18-Aug	20	54.14	53.82	0.85	7.82	7.85
25-Aug	10	40.51	38.90	0.68	7.73	7.90
01-Sep	10	24.43	22.00	0.85	7.85	7.99
08-Sep	70	18.16	18.14	0.86	7.79	7.78
15-Sep	200	18.96	18.93	0.87	7.67	7.69
22-Sep	50	28.06	28.06	0.88	7.76	7.75
29-Sep	10	31.32	31.04	0.93	7.78	7.83
06-Oct	20	31.80	31.65	0.94	7.81	7.82
13-Oct	10	23.67	23.42	0.85	7.73	7.73
20-Oct	10	28.69	27.48	0.90	7.75	7.72
27-Oct	10	30.69	30.17	0.91	7.74	7.72
03-Nov	10	23.56	22.97	0.66	7.77	7.74
10-Nov	10	42.13	41.00	0.85	8.13	8.00

EXHIBIT 2 ■ Continued

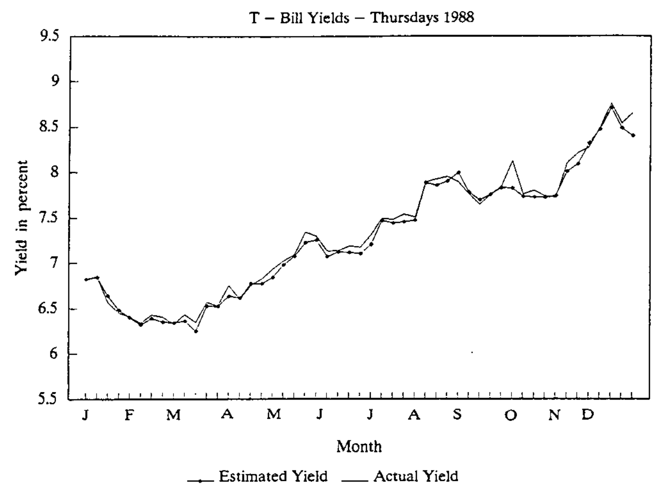
	Second-Order Model					
Date	Optimal tau	SD at Best tau	SD at tau = 40	R^2	Yield at Best tau	Yield at tau = 40
17-Nov	10	41.17	37.19	0.89	8.19	8.09
24-Nov	10	50.43	49.20	0.78	8.50	8.32
01-Dec	30	56.62	56.58	0.93	8.51	8.47
08-Dec	10	43.28	43.07	0.91	8.72	8.71
15-Dec	30	64.31	64.30	0.84	8.54	8.49
22-Dec	20	103.68	101.65	0.90	8.60	8.40

EXHIBIT 3 ■ T-Bill Yields ■ Mondays 1988



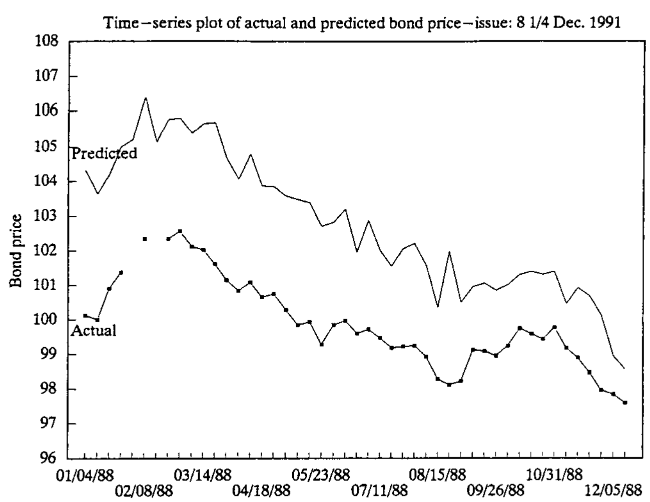
Note: The estimated yield is based on tau = 40

EXHIBIT 4 ■ T-Bill Yields ■ Thursdays 1988



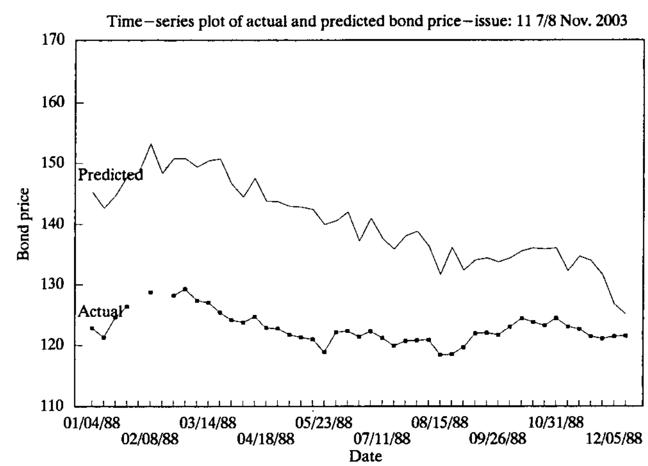
Note: The estimated yield is based on tau = 40

EXHIBIT 5 ■ Time-Series Plot of Actual and Predicted Bond Price ■ Issue: 8 1/4 December 1991



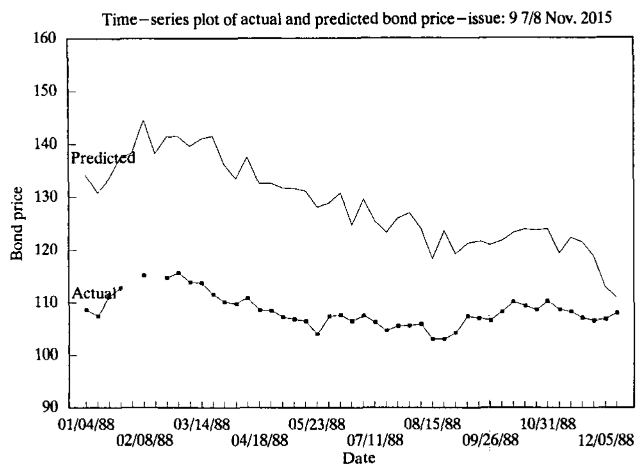
Note: The predicted bond price was based on tau = 40

EXHIBIT 6 ■ Time-Series Plot of Actual and Predicted Bond Price ■ Issue: 11 7/8 November 2003



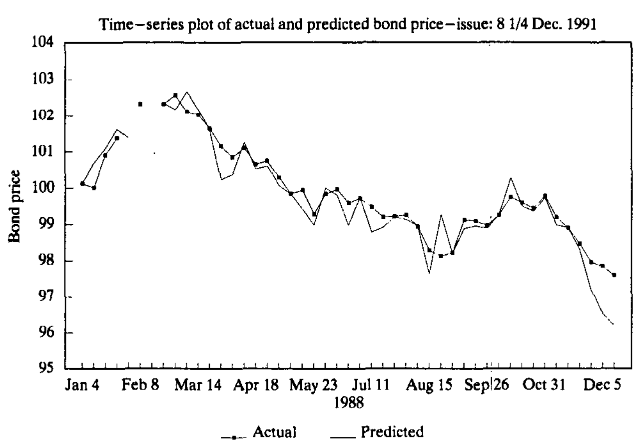
Note: The predicted bond price was based on tau = 40

EXHIBIT 7 ■ Time-Series Plot of Actual and Predicted Bond Price ■ Issue: 9 7/8 November 2015



Note: The predicted bond price was based on tau = 40

EXHIBIT 8 ■ Time-Series Plot of Actual and Predicted Bond Price ■ Issue: 8 1/4 December 1991

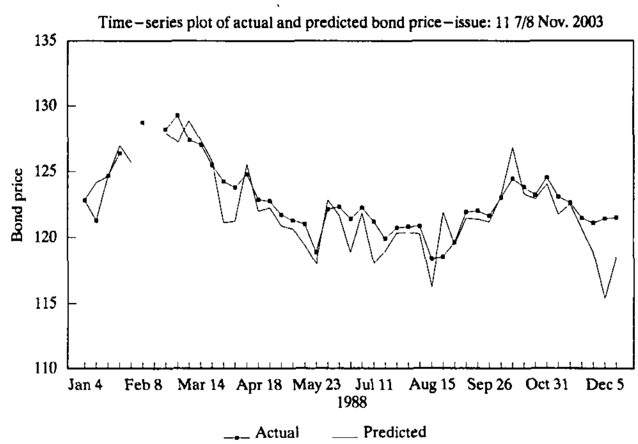


Note: The predicted bond price was based on tau = 40. The long–term component of the yield was multiplied by a factor implied by the term structure of the previous Thursday.

These findings are consistent with those of Brown and Dybvig [1986]. In all cases, however, there is a high correlation between actual and predicted bond prices, supporting the appropriateness of the model in predicting the decay of the yield on U.S. Treasury bonds, rather than its actual level. That is, although the time constant τ^* is optimal for fitting the T-bill yield curve, it does not account for the risk premium inherent in the term structure of the T-bonds. This is precisely the contribution of the CIR model, which allows for such differentials in interest rate risk premiums.

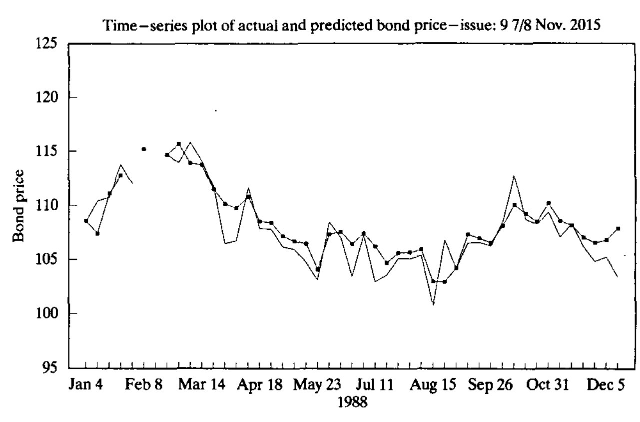
If we assume that the CIR parameters κ, λ, and σ^2 change through time so as to keep the ratio $[2\kappa/(\gamma + \kappa + \lambda)]$ either constant or near constant, we can approximate the current period's ratio with its lagged value, i.e., $[2\kappa/(\gamma + \kappa + \lambda)]_t \approx [2\kappa/(\gamma + \kappa + \lambda)]_{t+\delta}$ where δ represents a two-day interval, in our case. For this purpose, we estimate Equation (8) for every available Monday during 1988, using the parameters implied by the previous Thursday's data for the three bonds corresponding to Exhibits 5 through 7.[11] The

EXHIBIT 9 ■ Time-Series Plot of Actual and Predicted Bond Price ■ Issue: 11 7/8 November 2003



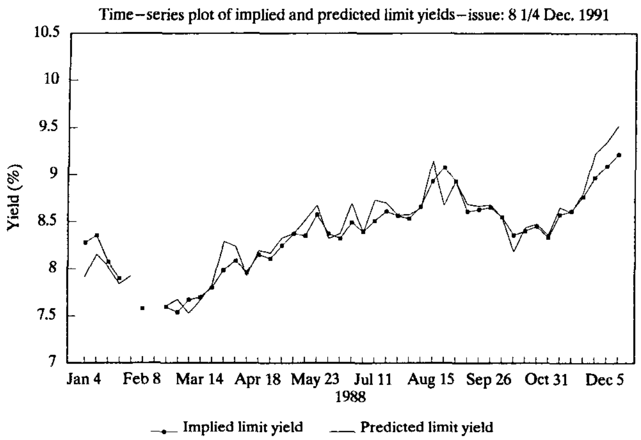
Note: The predicted bond price was based on tau = 40. The long–term component of the yield was multiplied by a factor implied by the term structure of the previous Thursday.

EXHIBIT 10 ■ Time-Series Plot of Actual and Predicted Bond Price ■ Issue: 9 7/8 November 2015



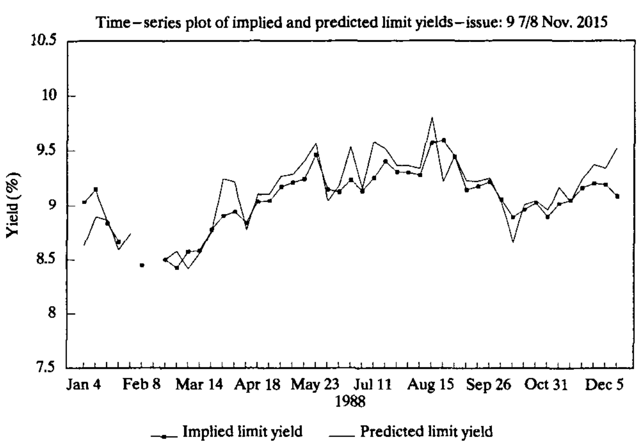
Note: The predicted bond price was based on tau = 40. The long–term component of the yield was multiplied by a factor implied by the term structure of the previous Thursday.

EXHIBIT 11 ■ Time-Series Plot of Implied and Predicted Limit Yields ■ Issue: 8 1/4 December 1991



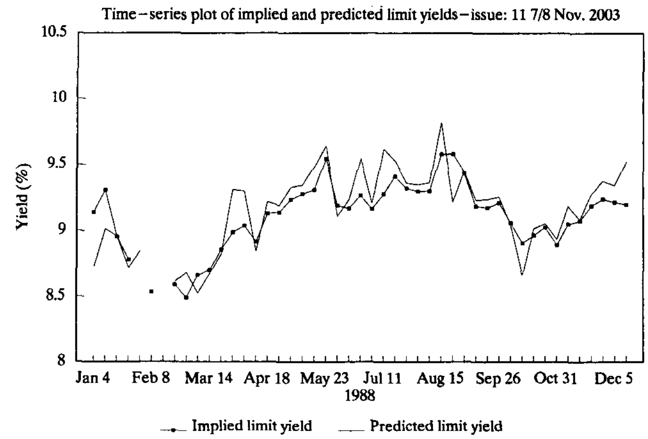
Note: The predicted limit yield is the product of the long–term component of the yield and a factor implied by the term structure of the previous Thursday.

EXHIBIT 12 ■ Time-Series Plot of Implied and Predicted Limit Yields ■ Issue: 11 7/8 November 2003



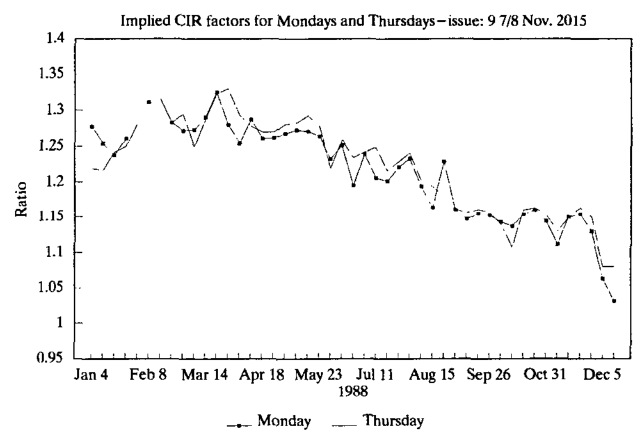
Note: The predicted limit yield is the product of the long–term component of the yield and a factor implied by the term structure of the previous Thursday.

EXHIBIT 13 ■ Time-Series Plot of Implied and Predicted Limit Yields ■ Issue: 9 7/8 November 2015



Note: The predicted limit yield is the product of the long–term component of the yield and a factor implied by the term structure of the previous Thursday.

EXHIBIT 14 ■ Implied CIR Factors for Mondays and Thursdays ■ Issue: 9 7/8 November 2015



Note: *The implied CIR factor is a ratio such that its product with the long–term value of the yield equals the asymptotic value of the yield.*

forecasted bond prices and corresponding yields are contrasted with their actual counterparts in Exhibits 8 through 13.

Exhibit 14 compares the implied CIR ratios $[2\kappa/(\gamma + \kappa + \lambda)]$ for Mondays and Thursdays for the long-term bond. The value of these ratios ranges from 1.03 to 1.33. The maximum likelihood estimates of Chen and Scott [1990] implied a ratio of 1.06 for the period 1980 to 1988. As Exhibit 14 shows, the ratios declined systematically, although not monotonically, which was in line with the flattening of the yield curve during 1988. As we would expect intuitively, a near-flat yield curve would imply very low risk premiums and a limit yield, $R(\infty)$, approximately equal to its long-term value θ.

The main point of Exhibit 14 is that although the CIR factor is not constant through time, at least it is not stochastic. This evidence is in line with the CIR methodology, which implies that the limiting value of the yield is independent of the state variable represented by the current rate of interest. The non-stochastic nature of this ratio allows us to approximate it with its lagged value, as explained earlier. Thursday's implied factor tracks the following Monday's factor fairly well.

We would intuitively expect the one-day time interval to yield better results, or the Friday-Monday factors to be even closer.

Turning to Exhibits 8 through 13, it is evident that the methodology proposed in this article yields better results. In all three cases, the forecasted values according to Equation (8) are closer to their actual counterparts than those forecasted by the Nelson and Siegel [1987] methodology as expressed in Equation (7). The predictions tend to overshoot the actuals, but not by as much as those reported by Nelson and Siegel. The magnitude of overshooting documented here may be related to the tax effect reasoned by Nelson and Siegel, but such a reckoning seems unlikely, especially as the overshooting is not systematic.

CONCLUSIONS

Our simple approach to pricing long-term default-free bonds combines the Cox, Ingersoll, and Ross [1985] theoretical model with the yield curve approximations suggested by Nelson and Siegel [1987] and Siegel and Nelson [1988]. The resulting term structure of interest rates decomposes the risk premium into two parts. The dominant term of the risk premium depends on risk-return preference parameters that cannot be determined from the current rate of interest and cannot be priced from the term structure of T-bills. The remaining risk premium is related to the curvature (rate of decay) of the yield curve, which can easily be determined from the term structure of T-bills.

The former part of the risk premium involves the CIR limiting value of the yield and its parameters κ, γ, λ, and θ. The latter risk premium is approximated from the Nelson and Siegel [1987] and Siegel and Nelson [1988] curve-fitting models.

According to the fundamental pricing equation developed, the term structure of T-bills will price long-term bonds correctly only if the CIR ratio $[2\kappa/(\gamma + \kappa + \lambda)]$ is equal to unity or, equivalently, if the T-bills and T-bonds differ only in maturity. The empirical evidence presented clearly demonstrates that this is not the case, as we would have expected. By approximating the current period's ratio with its lagged value, however, the forecasted bond prices track the actuals with significant accuracy.

ENDNOTES

The authors acknowledge helpful comments from George C. Philippatos, Lee Sarver, and the participants in the Eighth International Symposium on Cash, Treasury, and Working Capital Management.

[1]Power [1991] finds that the Fisher [1966] and Nelson-Siegel [1987] estimators dominate the Cohen-Kramer-Waugh [1966] parametric estimators, especially when the coupon rate is included. The Power [1990] non-parametric estimator is the clear dominant estimator only when the coupon rate is not included.

[2]This approach is analogous to the concept of the present value of an annuity. In a way, the present value of an annuity from period 1 to period m can be thought of as the difference between the present value of a perpetuity from period 1 and a second perpetuity from period m. See Brealey and Myers [1991, p. 34] for more details and an illustration of this concept.

[3]The yield R denotes the discount rate on a default-free discount bond such that $\$1 = Pe^{mR}$, or $R = -\log(P)/m$, where P is the current price of the bond and m is the time to maturity expressed as a fraction of a year.

[4]Labeling the long-term component of the yield as equivalent to CIR's steady state mean θ is not of critical importance in this article. As will be seen later, the interest rate risk premium has two components: one that depends on risk-return preference parameters that cannot be determined from the current level of interest, and the other related to the term structure of T-bills. In other words, while T-bills and T-bonds may share the same steady state mean interest rate, θ, they may differ in terms other than merely time to maturity. It is in this sense that we relate the first term of the risk premium to CIR's limiting value of the yield, rather than θ.

[5]The proof follows from the fact that the last term in Equation (7) tends to zero as $m \rightarrow \infty$ and that the limiting value of R(m) equals the limiting value of r(m) under Assumptions A.1 and A.2.

[6]Cox, Ingersoll, and Ross [1985, p. 394] provide a more formal explanation as to why the return on very short-term bonds becomes certain.

[7]Nelson and Siegel [1987] omit these observations also. Their conjecture is that the relatively large transaction

costs over a short period of time increased these yields. This hypothesis is consistent with the findings of Choi and Jen [1991], who reported a significant relationship between the turn-of-the-year seasonality, often attributed to tax considerations, and the interest rate risk premium.

[8]The median value of τ reported by Nelson and Siegel [1987] was equal to 50. The yield curve was upward-sloping in 1987, became flat in 1988, and was inverted in 1989, signaling the possibility of a recession. For a detailed discussion on the implications of the yield curve for economic conditions, see McCallum [1989].

[9]The discount rate applicable to the bond payments is the yield to maturity implied by the quoted asked price of the bond.

[10]It would be interesting to consider pure discount bonds because, like bills, they have no reinvestment rate risk; they bear only liquidity risk.

[11]Although it is not possible to identify the parameters κ, γ, and λ separately in Equation (8), we can estimate the ratio $[2\kappa/(\gamma + \kappa + \lambda)]$, assuming we know the remaining parameters θ, β_1, β_2, and τ^*. The latter set of parameters can be estimated by fitting Equation (7) to Treasury bills.

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